Journal homepage www.jzs.uos.edu.krd
Journal of Zankoi Sulaimani
Part-A- (Pure and Applied Sciences)

Some Properties of Binormal Operators

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Article info

Original: 16 Apr. 2015
Revised: 25 May 2015
Accepted: 31 May 2015
Published online:
20 Dec. 2015

Key Words:
Normal operator , n-Normal operator, Binormal operator .

Abstract

In this paper we have introduced some classes of operators like class of normal operators and n-binormal operators and n-isometry operators on an infinite Hilbert space H. We have given some basic properties of these classes of operators. In general, every normal operator is not necessarily an isometry operator and we studied the relationship between normal with n-isometry and normal with n-binormal .We concluded the relationship between n-isometry with n-binormal.

Introduction

Here and hereafter, $B(H)$ and $NB(H)$ denote, respectively, to the algebra of all bounded and class of normal linear operators acting on a complex Hilbert space H .An operator $T \in B(H)$ is called normal operator if $TT^* = T^*T$, n-normal if $T^nT^* = T^*T^n$, binormal if T^*T commutes with TT^*, n- binormal if $T^*T^nT^nT^* = T^nT^*T^*T^n$, isometry if $T^*T=I$, 2-isometry if $T^{*2}T^2 - 2T^*T + I = 0$ and n-isometry if $\sum_{k=0}^{n}(-1)^k \binom{n}{k} T^{*n-k}T^{n-k} = 0$.

The properties of binormal, n-binormal, isometry and n-isometry operators and operators related with them were extensively studied by many authors Panayappan and Sivamani(2012) studied n-binormal operators, Alzuraiqi, Patel(2010) On n-Normal Operators, Riyadh Rustam Al-Mosawi and Hadeel Ali Hassan(2014) A Note On Normal and n-Normal Operators, Jibril, A.A.S. (2008) On n-power normal operators, Jibril (2010) On operators for which $T^{*2}T^2 = (T^*T)^2$, Panayappan, (2012) On n-power class (Q) operators, Naoum and Nassir (2007) Pseudo-normal operators.

The Main Results

For simplicity, we cursor $IB(H)$ to denote the algebra of invertible linear bounded operators acting on Hilbert space and we cursor $INB(H)$ if the operator is an invertible normal.

Proposition 2.1: If $T \in INB(H)$ is an operator, then

$$\left((T^*T^{-1})^*(T^*T^{-1})\right)^n = \ (T^*T^{-1})^{*^n}(T^*T^{-1})^n = I.$$

Proof: We prove by using Mathematical Induction.

For $n = 2$ and since $T \in INB(H)$,

$$\left((T^*T^{-1})^*(T^*T^{-1})\right)^2 = (T^*T^{-1})^*(T^*T^{-1})(T^*T^{-1})^*(T^*T^{-1})$$
$$= T^{*^{-1}}TT^*T^{-1}T^{*^{-1}}TT^*T^{-1} = T^{*^{-1}}T^*TT^{-1}T^{*^{-1}}T^*TT^{-1} = I.$$

Now,

$$(T^*T^{-1})^{*^2}(T^*T^{-1})^2 = (T^*T^{-1})^*(T^*T^{-1})^*(T^*T^{-1})(T^*T^{-1}) = T^{*^{-1}}TT^{*^{-1}}TT^*T^{-1}T^*T^{-1}$$
$$= T^{*^{-1}}\ T\ T^{*^{-1}}T^*TT^{-1}T^*T^{-1} = T^{*^{-1}}\ T\ IT^*T^{-1} = T^{*^{-1}}\ \ T^*TT^{-1} = I.$$

and

$$\left((T^*T^{-1})^*(T^*T^{-1})\right)^2 = \ (T^*T^{-1})^{*^2}(T^*T^{-1})^2 = I.$$

Suppose that the statement is true if $n = k$, i.e.

$$\left((T^*T^{-1})^*(T^*T^{-1})\right)^k = \ (T^*T^{-1})^{*^k}(T^*T^{-1})^k = I.$$

For $n = k + 1$, Since $T \in INB(H)$,

$$\left((T^*T^{-1})^*(T^*T^{-1})\right)^{k+1} = ((T^*T^{-1})^*(T^*T^{-1}))\left((T^*T^{-1})^*(T^*T^{-1})\right)^k.$$

Since the statement is true for $n = K$,

$(T^{*^{-1}}T^*TT^{-1})(T^*T^{-1})^{*^k}(T^*T^{-1})^k = I(T^*T^{-1})^{*^k}(T^*T^{-1})^k = (T^*T^{-1})^{*^k}I(T^*T^{-1})^k =$

$(T^*T^{-1})^{*^k}(T^{*^{-1}}T^*TT^{-1})(T^*T^{-1})^k = (T^*T^{-1})^{*^k}(T^*T^{-1})^*(T^*T^{-1})(T^*T^{-1})^k =$

$(T^*T^{-1})^{*^{k+1}}(T^*T^{-1})^{k+1} = I.$

This means that the statement is true if $= k + 1$.

Hence $\left((T^*T^{-1})^*(T^*T^{-1})\right)^n = \ (T^*T^{-1})^{*^n}(T^*T^{-1})^n = I.$ ■

Remark 2.1 Clearly, if $T \in INB(H)$, then $T^* \in INB(H)$ and $T^{-1} \in INB(H)$. (see Gheondea (2009)).

Theorem2.1: If $T \in INB(H)$ is an operator, then T^*T^{-1} is a n-binormal operator.

Proof: We prove by using Mathematical Induction
For $n = 1$

$(T^*T^{-1})(T^*T^{-1})^*(T^*T^{-1})^*(T^*T^{-1}) = T^*T^{-1}T^{*^{-1}}TT^{*^{-1}}TT^*T^{-1}$

$= T^*T^{-1}T^{*^{-1}}TT^{*^{-1}}T^*TT^{-1} = T^*T^{-1}T^{*^{-1}}TII = \ T^*T^{*^{-1}}T^{-1}T = I$ (because T^{-1} is normal) .

Now,

$$(T^*T^{-1})^*(T^*T^{-1})(T^*T^{-1})(T^*T^{-1})^* = {T^*}^{-1}TT^*T^{-1}T^*T^{-1}{T^*}^{-1}T = {T^*}^{-1}T^*TT^{-1}T^*T^{-1}{T^*}^{-1}T$$
$$= IIT^*T^{-1}{T^*}^{-1}T = T^*{T^*}^{-1}T^{-1}T = I.$$

So

$(T^*T^{-1})(T^*T^{-1})^*(T^*T^{-1})^*(T^*T^{-1}) = (T^*T^{-1})^*(T^*T^{-1})(T^*T^{-1})(T^*T^{-1})^*$.

Hence $T^*T^{-1}$is binormal operator.
Suppose that the statement is true for $n = k$, i.e.

$$(T^*T^{-1})^k(T^*T^{-1})^*(T^*T^{-1})^*(T^*T^{-1})^k = (T^*T^{-1})^*(T^*T^{-1})^k(T^*T^{-1})^k(T^*T^{-1})^*.$$

For $n = k + 1$
$(T^*T^{-1})^{k+1}(T^*T^{-1})^*(T^*T^{-1})^*(T^*T^{-1})^{k+1} = (T^*T^{-1})(T^*T^{-1})^k(T^*T^{-1})^*(T^*T^{-1})^*(T^*T^{-1})^k(T^*T^{-1})$.
Since the statement is true for $n = k$
$(T^*T^{-1})(T^*T^{-1})^*(T^*T^{-1})^k(T^*T^{-1})^k(T^*T^{-1})^*(T^*T^{-1}) =$
$(T^*T^{-1})^*(T^*T^{-1})(T^*T^{-1})^k(T^*T^{-1})^k(T^*T^{-1})(T^*T^{-1})^* = (T^*T^{-1})^*(T^*T^{-1})^{k+1}(T^*T^{-1})^{k+1}(T^*T^{-1})^* =$
$(T^*T^{-1})^{k+1}(T^*T^{-1})^*(T^*T^{-1})^*(T^*T^{-1})^{k+1} = (T^*T^{-1})^*(T^*T^{-1})^{k+1}(T^*T^{-1})^{k+1}(T^*T^{-1})^*$ (because $\in$ $INB(H)$, $T^*T^{-1} \in INB(H)$).
Therefore, the Statement is true if $n = k + 1$. Hence the statement is true for any $n \in N$, i.e.
$(T^*T^{-1})^n(T^*T^{-1})^*(T^*T^{-1})^*(T^*T^{-1})^n = (T^*T^{-1})^*(T^*T^{-1})^n(T^*T^{-1})^n(T^*T^{-1})^*$.

Therefore, T^*T^{-1} is a n-binormal operator ■

It is well-known that every normal operator not necessarily be an isometry operator.

Example :$T = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} \rightarrow T$ is normal but

$$T^*T = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}\begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 49 & 0 \\ 0 & 49 \end{pmatrix} \neq I$$

T is normal but not isometry

Theorem2.2: Let $T \in IB(H)$ is an operator. Then T is a normal operator iff T^*T^{-1} is an isometry operator.

Proof: Suppose that T normal, this means that

$(T^*T^{-1})^*(T^*T^{-1}) = {T^*}^{-1}TT^*T^{-1} = {T^*}^{-1}T^*TT^{-1} = I.$

Hence T^*T^{-1} is isometry. Suppose that T^*T^{-1} is isometry. Then $(T^*T^{-1})^*(T^*T^{-1}) = {T^*}^{-1}TT^*T^{-1} = I$

. *Hence* $T^*{T^*}^{-1}TT^*T^{-1} = T^*$. *So* $TT^*T^{-1}T = T^*T$. *Therefore* $TT^* = T^*T$.

So we have got T is normal. ■

Theorem2.3: If $T \in INB(H)$ is an operator, then T^*T^{-1} 2-isometry iff T^*T^{-1} isometry

Proof: Suppose that T^*T^{-1} be an 2-isometry, then

$$(T^*T^{-1})^{*^2}(T^*T^{-1})^2 - 2(T^*T^{-1})^*(T^*T^{-1}) + I = 0. \text{ So } ((T^*T^{-1})^*(T^*T^{-1}) - I)^2 = 0.$$

Therefore $(T^*T^{-1})^*(T^*T^{-1}) - I = 0$. Hence $(T^*T^{-1})^*(T^*T^{-}) = I$. So we have got T^*T^{-1} isometry.

Suppose that T^*T^{-1} is an isometry, then $(T^*T^{-1})^*(T^*T^{-1}) = I$. So $(T^*T^{-1})^*(T^*T^{-1}) - I = 0$. Therefore $((T^*T^{-1})^*(T^*T^{-1}) - I)^2 = 0$. Hence $(T^*T^{-1})^{*^2}(T^*T^{-1})^2 - 2(T^*T^{-1})^*(T^*T^{-1}) + I = 0$.

So T^*T^{-1} 2-isometry. ■

Theorem2.4: Let $T \in IB(H)$ is an operator. Then T is a normal operator iff T^*T^{-1} is n-isometry operator.

Proof: Suppose that T normal. Hence

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(T^*T^{-1})^{*^{n-k}}(T^*T^{-1})^{n-k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(T^*T^{-1})^{*^{n-k}}(T^*T^{-1})^{n-k}I^K =$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}((T^*T^{-1})^*(T^*T^{-1}))^{n-k}I^K = ((T^*T^{-1})^*(T^*T^{-1}) - I)^n = (T^{*^{-1}}TT^*T^{-1} - I)^n =$$

$$(T^{*^{-1}}T^*TT^{-1} - I)^n = 0.$$

Hence T^*T^{-1} n-isometry.

Suppose that T^*T^{-1} n-isometry. Hence

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(T^*T^{-1})^{*^{n-k}}(T^*T^{-1})^{n-k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(T^*T^{-1})^{*^{n-k}}(T^*T^{-1})^{n-k}I^k =$$

$$((T^*T^{-1})^*(T^*T^{-1}) - I)^n = T^{*^{-1}}TT^*T^{-1} - I = 0.$$

$T^{*^{-1}}TT^*T^{-1} = I$. Thus $TT^* = T^*T$, which is implies that T *is* normal. ■

Corollary 2.1: Let $T \in IB(H)$ is an operator. Then T^*T^{-1} is an isometry operator iff T^*T^{-1} is n-isometry operator.

Proof: Suppose that T^*T^{-1} is an isometry operator, then

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(T^*T^{-1})^{*^{n-k}}(T^*T^{-1})^{n-k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(T^*T^{-1})^{*^{n-k}}(T^*T^{-1})^{n-k}I^k =$$

$((T^*T^{-1})^*(T^*T^{-1}) - I)^n = 0$ (because T^*T^{-1} is an isometry operator). Hence T^*T^{-1} is n-isometry operator.

Suppose that T^*T^{-1} is n-isometry operator, then

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(T^*T^{-1})^{*^{n-k}}(T^*T^{-1})^{n-k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(T^*T^{-1})^{*^{n-k}}(T^*T^{-1})^{n-k}I^k =$$

$$((T^*T^{-1})^*(T^*T^{-1}) - I)^n = (T^*T^{-1})^*(T^*T^{-1}) - I = 0.$$

Hence

$(T^*T^{-1})^*(T^*T^{-1}) = I$. *So* T^*T^{-1} is an isometry operator ■

Corollary 2.2: Let $T \in IB(H)$ is an operator . If $T^{*}T^{-1}$ is an isometry operator, then $T^{*}T^{-1}$ is binormal operator.

Proof: Since $T^{*}T^{-1}$ is an isometry operator, by theorem (2.3), T is normal operator.

Since T is normal operator, by theorem (2.1), $T^{*}T^{-1}$ is binormal operator ■

Corollary 2.3: Let $T \in IB(H)$ is an operator . If $T^{*}T^{-1}$ is n- isometry operator, then $T^{*}T^{-1}$ is n-binormal operator.

Proof: Since $T^{*}T^{-1}$ is n-isometry operator , by theorem (2.4), we have got T is normal operator

Since T is normal operator, by theorem (2.1), we have got $T^{*}T^{-1}$ is n-binormal operator ■

Acknowledgements

The author would like to thank Professor Dr. Riyadh Rustam Al-Mosawi/Mathematics department/Thi-Qar University/Iraq and Dr.Shaker Abdul-Rahamn Alzoriqi/Yamen for a careful reading of the manuscript and for many helpful comments and suggestions.

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